BOUNDING CRYSTALLINE TORSION FROM ÉTALE TORSION

OFER GABBER AND SHIZHANG LI

Abstract. In this note, we prove that given a smooth proper family over a p-adic ring of integers, one gets a control of its crystalline torsion in terms of its étale torsion, the cohomological degree, and the ramification. Our technical core result is a boundedness result concerning annihilator ideals of u^∞-torsion in Breuil–Kisin prismatic cohomology, which might be of independent interest.

Contents

1. Introduction

Let p be a prime. Let $\mathcal{O}_K$ be a complete DVR of mixed characteristic $(0, p)$ with perfect residue field k. Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$. We are interested in the interplay between two torsion phenomena associated with $\mathcal{X}$: the étale torsion $\mathrm{H}^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p)_{\mathrm{tors}}$ and the crystalline torsion $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$. In Bhatt–Morrow–Scholze's first paper [BMS18], one learns the following:

Theorem 1.1 ([BMS18, Theorem 1.1.(ii)])**.** *There is an inequality*

$$\mathrm{length}\left(\mathrm{H}^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p)_{\mathrm{tors}}\right) \leq \mathrm{length}\left(\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}\right).$$

In particular, if $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}} = 0$*, then* $\mathrm{H}^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p)_{\mathrm{tors}} = 0$.

It is natural to wonder about the converse question: if $\mathrm{H}^i_{\acute{e}t}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_p)_{\mathrm{tors}} = 0$, then what can we say about $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$? The Künneth formula and examples in [BMS18, Section 2] shows that one cannot get any bound on the length of the $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$. In this paper we show that one can get a bound of the exponent of the $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}}$, defined as the smallest natural number m such that $p^m \cdot \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W(k))_{\mathrm{tors}} = 0$.

Theorem 1.2 (= Theorem 4.8)**.** *There is a constant $c(e, i)$ depending only on the ramification index $e = v_K(p)$ and the cohomological degree $i > 0$, such that there is an inequality*

$$\exp(\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W)_{\mathrm{tors}}) \leq \exp(\mathrm{H}^i_{\acute{e}t}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}}) + c(e, i).$$

Our technical tool is a generalization of some results in [LL23], concerning the annihilator ideal of u^∞-torsion in prismatic cohomology of $\mathcal{X}$. This is the content of our Section 2. In Section 4, we give some applications of the bound of aforementioned annihilator ideals, and end with a proof of the above theorem.

Notations and Conventions. Let k be a perfect field of characteristic p, let $W = W(k)$ be its (p-typical) Witt ring. Denote $\mathfrak{S} := W[\![u]\!]$ equipped with (u, p)-adically continuous Frobenius $\varphi\colon \mathfrak{S} \to \mathfrak{S}$ such that $\varphi|_W$ is the usual Witt vector Frobenius and $\varphi(u) = u^p$. Lastly let $E(u) \in \mathfrak{S}$ be an Eisenstein polynomial of degree e.

2. Some commutative algebra arguments

Throughout this section, we shall consider the following situation.

Situation 2.1. Let $J \subset \mathfrak{S}$ be an ideal and let $j \in \mathbb{N}$, satisfying

(1) the ideal J is cofinite, namely $(p,u)^N \subset J$ for some N; and
(2) we have a containment relation $E^j \cdot J \subset \varphi(J) \cdot \mathfrak{S}$.

In this situation, let us denote $J + (p) = (p, u^\sigma)$ and $J + (u) = (u, p^\rho)$. It is easy to see that $\sigma \leq \lfloor \frac{e \cdot j}{p-1} \rfloor$, see for instance the proof of [LL23, Corollary 3.4].

The aim of this section is to give explicit estimate of ρ in terms of e and j.

2.1. Argument one.

In this subsection, we present the first argument.

Notation 2.2. Let $c(a,b) \coloneqq \min\{c \in \mathbb{N} \mid p^c \in (u^a, E^b)\}$.

Lemma 2.3. *We have that $c(a,b) \leq \lceil \frac{a}{e} \rceil + b - 1$.*

Proof. By assumption $E = u^e + p \cdot \text{unit}$, so $p \in (u^e, E)$. More generally we have $p^{x+y-1} \in (u^{ex}, E^y)$. □

Lemma 2.4. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. Suppose that $J + (u^a) \subset (u^a, p^N)$, then $J \subset (u^A, p^{\max\left(0, N - c(A,j)\right)})$ for any natural number $A \leq pa$.*

Proof. In the ring $R \coloneqq \mathfrak{S}/u^A$, we have

$$p^{c(A,j)} \cdot JR \subset E^j \cdot JR \subset \varphi(J) \cdot R \subset (p^N).$$

Our claim follows from the fact that the sequence (u,p) is $\mathfrak{S}$-regular. □

Proposition 2.5. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. Let $a_1, a_2, \ldots, a_n$ be a sequence of integers satisfying*

(1) $a_0 = 1$;
(2) $a_i \leq p \cdot a_{i-1}$;
(3) $a_n > \frac{e \cdot j}{p-1}$.

Then $\rho \leq \sum_{i=1}^n c(a_i, j)$.

In particular, if $e \cdot j < p^n(p-1)$, then we may choose $a_i = p^i$ for $i \leq (n-1)$ and $a_n = \lfloor \frac{e \cdot j}{p-1} \rfloor + 1$, hence $\rho \leq \sum_{i=1}^{n-1} \lfloor \frac{p^i}{e} \rfloor + \lfloor \frac{\lfloor \frac{e \cdot j}{p-1} \rfloor + 1}{e} \rfloor + nj$.

Proof. The second sentence follows from the first one and Lemma 2.3 as $\lceil x \rceil - 1 < x$. To see the first sentence: Let $J + (u) = (u, p^\rho)$, and assume to the contrary that $\rho > \sum_{i=1}^n c(a_i, j)$. Then applying Lemma 2.4, we see that $J + (u^{a_1}) \subset (u^{a_1}, p^{\rho - c(a_1, j)})$. Applying Lemma 2.4 again, we see that $J + (u^{a_2}) \subset (u^{a_2}, p^{\rho - c(a_1,j) - c(a_2,j)})$. Repeating the above, we finally see that $J + (u^{a_n}) \subset (u^{a_n}, p^{>0})$. But this contradicts to the fact that $J + (p) = (p, u^\sigma)$ with $\sigma \leq \frac{e \cdot j}{p-1} < a_n$. □

2.2. Argument two.

In this subsection, we present the second argument. Throughout the subsection, let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1.

Lemma 2.6. *Let $r \in [0, \infty)$ be a real number, the following map*

$$v_r \colon \mathfrak{S} \setminus \{0\} \to \mathbb{R},\ v_r(\sum_i a_i u^i) \coloneqq \min\{\mathrm{ord}_p(a_i) + i \cdot r\}$$

defines an additive valuation.

Proof. It is easy to check that minimum is always attained, one can check the triangle inequality

$$v_r\left((\sum_i a_i u^i) + (\sum_i b_i u^i)\right) \geq \min\left(v_r(\sum_i a_i u^i), v_r(\sum_i b_i u^i)\right)$$

using the definition. Lastly we need to check multiplicativity:

$$v_r\left((\sum_i a_i u^i)\cdot(\sum_i b_i u^i)\right) = v_r(\sum_i a_i u^i) + v_r(\sum_i b_i u^i).$$

One checks directly that the multiplicativity holds true if one of the power series is just a monomial. Now let $\alpha \coloneqq \min\{i \in \mathbb{N} \mid \mathrm{ord}_p(a_i) + i\cdot r = v_r(\sum_i a_i u^i)\}$ and $\beta \coloneqq \min\{i \in \mathbb{N} \mid \mathrm{ord}_p(b_i) + i\cdot r = v_r(\sum_i b_i u^i)\}$. Using the definition, one checks that

$$v_r\left((\sum_{i\geq\alpha} a_i u^i)\cdot(\sum_{i\geq\beta} b_i u^i)\right) = v_r(\sum_i a_i u^i) + v_r(\sum_i b_i u^i).$$

Finally, by combining

- the case of one of the power series being a monomial;
- the decompositions $\sum_i a_i u^i = \sum_{i<\alpha} a_i u^i + \sum_{i\geq\alpha} a_i u^i$ and $\sum_i b_i u^i = \sum_{i<\beta} b_i u^i + \sum_{i\geq\beta} b_i u^i$ of the two power series;
- the above equality; and
- the triangle inequality,

one arrives at the multiplicativity statement which finishes the proof. □

One may view the ring $\mathfrak{S}$ as the analytic functions bounded by 1 on the open unit disc $\mathbb{D}^{\circ}_{W[1/p]}$, then the valuation v_r corresponds to the Gauss norm on the radius p^{-r} disc (the absolute value is normalized so that $|p| = p^{-1}$). Notice that for $r > 0$, we can take a rational number $s \in (0, r]$, so the said Gauss norm is a rank 1 point on the closed disc of radius p^{-s} around 0. Therefore, we may view it as a rank 1 point on the open unit disc, giving rise to a norm on $\mathfrak{S}[1/p]$ whose restriction to $\mathfrak{S}$ is bounded by 1.

Notations 2.7. For any co-finite ideal $I \subset \mathfrak{S}$, let $f_I(r) \coloneqq v_r(I)$, viewed as a function $f_I \colon [0,\infty) \to \mathbb{R}_{\geq 0}$. Let $I^{\mathrm{mon}} \coloneqq$ the ideal generated by $\{a_i u^i \mid \sum_i a_i u^i \in I\}$.

Namely for every power series in I, we extract out all of its monomial terms, then we use all these monomial terms of all elements in I to generate a (most likely larger) ideal. Note that I^{mon} is generated by finitely many monomial terms as $\mathfrak{S}$ is Noetherian.

Lemma 2.8. *Let $I \subset \mathfrak{S}$ be a co-finite ideal, we have natural numbers σ and ρ satisfying $I + (p) = (p, u^{\sigma})$ and $I + (u) = (u, p^{\rho})$. Then the function f_I satisfies the following:*

(1) We have an equality $f_I = f_{I^{\mathrm{mon}}}$;
(2) The function f_I is concave and continuous;
(3) The function f_I is piecewise linear, on each interval it is given by $a\cdot r + b$ with both a and b natural numbers;
(4) There exists an $\epsilon > 0$, such that

$$f_I(r) = \begin{cases} \sigma\cdot r, & r \in [0,\epsilon], \\ \rho, & r \in [1/\epsilon, \infty). \end{cases}$$

(5) We have an equality $f_{\varphi(I)}(r) = f_I(p\cdot r)$.

Proof. (1) and (5) follows from the definition of v_r. Our assumption implies that

$$I^{\mathrm{mon}} = (p^{\rho}, a_1\cdot u, a_2\cdot u^2, \ldots, a_{\sigma-1}u^{\sigma-1}, u^{\sigma}),$$

where $\mathrm{ord}_p(a_i) > 0$ (and a_i is allowed to be 0). For each of the generators above, if we look at their v_r as a function in r, we simply get a linear function with a natural number slope. The function $f_I = f_{I^{\mathrm{mon}}}$ is minimum of the above collection of linear functions, this immediately gives us (2) and (3). Using (1) and the definition of v_r, we also see that $v_r(I) = v_r(u^{\sigma})$ if r is sufficiently near 0 and $v_r(I) = v_r(p^{\rho})$ if $r \gg 0$, which proves (4). □

Lemma 2.9. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. We have*

(1) *the function* $g(r) \coloneqq v_r(E^j) = \min\Big((e \cdot j) \cdot r, j\Big)$*; and*
(2) *an inequality* $f_J(p \cdot r) \leq f_J(r) + g(r)$.

Proof. (1) easily follows from our assumption on the degree e Eisenstein polynomial E. (2) follows from the assumption $E^j \cdot J \subset \varphi(J) \cdot \mathfrak{S}$ and Lemma 2.8.(5). $\square$

Lemma 2.10. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. Define a piecewise linear function*

$$h(r) = \begin{cases} \sigma \cdot r, & r \in [0, \frac{p \cdot j}{\sigma \cdot (p-1)}] \\ \frac{\sigma}{p} \cdot r + j, & r \in [\frac{p \cdot j}{\sigma \cdot (p-1)}, \frac{p^2 \cdot j}{\sigma \cdot (p-1)}] \\ \frac{\sigma}{p^2} \cdot r + 2 \cdot j, & r \in [\frac{p^2 \cdot j}{\sigma \cdot (p-1)}, \frac{p^3 \cdot j}{\sigma \cdot (p-1)}] \\ \cdots \\ \frac{\sigma}{p^n} \cdot r + n \cdot j, & r \in [\frac{p^n \cdot j}{\sigma \cdot (p-1)}, \frac{p^{n+1} \cdot j}{\sigma \cdot (p-1)}] \\ \cdots \end{cases}.$$

Then we have $f_J(r) \leq h(r)$.

We leave it to the readers to check that the $h(r)$ above is continuous, concave and increasing.

Proof. Let us check inductively on each interval that $f_J(r) \leq h(r)$. For the first interval $[0, \frac{p \cdot j}{\sigma \cdot (p-1)}]$, we need to show $f_J(r) \leq \sigma \cdot r$, this follows from Lemma 2.8.(2)-(4). Now we prove the induction step, so we assume that $f_J(x) \leq h(x)$ whenever $x \in [0, \frac{p^n \cdot j}{\sigma \cdot (p-1)}]$ and let $r \in [\frac{p^n \cdot j}{\sigma \cdot (p-1)}, \frac{p^{n+1} \cdot j}{\sigma \cdot (p-1)}]$. Our assumption on r implies that $f_J(\frac{r}{p}) \leq h(\frac{r}{p}) = \frac{\sigma}{p^{n-1}} \cdot \frac{r}{p} + (n-1) \cdot j$. By Lemma 2.9, we see that

$$f_J(r) \leq f_J(\frac{r}{p}) + j \leq \frac{\sigma}{p^{n-1}} \cdot \frac{r}{p} + (n-1) \cdot j + j = \frac{\sigma}{p^n} \cdot r + n \cdot j = h(r).$$

$\square$

Lemma 2.11. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. Then $f_J(r) = \rho$ whenever $r \geq \frac{p \cdot j}{p-1}$.*

Proof. Let us denote by $f'_J(r)$ the left derivative of $f_J(r)$, this is a piecewise constant, decreasing, eventually 0 function, which takes values in natural numbers, thanks to Lemma 2.8.(2)-(4). Therefore all we need to show is that $f'_J(r) = 0$ for $r > \frac{p \cdot j}{p-1}$. Now Lemma 2.9 implies that $f'_J(r) \cdot (r - \frac{r}{p}) \leq f_J(r) - f_J(\frac{r}{p}) \leq j$. Therefore $f'_J(r) < 1$ and is a natural number, hence must be 0. $\square$

Proposition 2.12. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. If $e \cdot j \leq p^n(p-1)$, then we have $\rho \leq (\frac{\sigma}{p^{n-1}(p-1)} + n) \cdot j \leq (\frac{\lfloor \frac{e \cdot j}{p-1} \rfloor}{p^{n-1}(p-1)} + n) \cdot j$.*

Proof. By Lemma 2.11, we have $\rho = f_J(\frac{p \cdot j}{p-1})$. Since $\sigma \leq \lfloor \frac{e \cdot j}{p-1} \rfloor \leq p^n$, we see that $\frac{p \cdot j}{p-1} \leq \frac{p^{n+1} \cdot j}{\sigma \cdot (p-1)}$ (and we only need to prove the first inequality). Now by Lemma 2.10, we have

$$\rho = f_J(\frac{p \cdot j}{p-1}) \leq h(\frac{p \cdot j}{p-1}) \leq \frac{\sigma}{p^n} \cdot \frac{p \cdot j}{p-1} + n \cdot j = (\frac{\sigma}{p^{n-1}(p-1)} + n) \cdot j.$$

$\square$

2.3. **Conclusions.** Let us first extract a concrete estimate of ρ in a special case.

Proposition 2.13. *Let $J \subset \mathfrak{S}$ be as in Situation 2.1, with $j = 1$, and let $n \in \mathbb{N}$.*
(1) *If $p \neq 2$ and $e < p^n(p-1)$, then $\rho \leq n$.*
(2) *If $p = 2$ and $e < 2^n$, then $\rho \leq (n+1)$.*

Note that when $e \leq (p-1)$, our statement follows from the proof of [LL23, Proposition 3.5]. So in the proof below, we always assume further that $e > (p-1)$, in particular $n \geq 1$.

Proof. First let us assume that $p \neq 2$. Suppose that $e < p^{n-1}(p-1)^2$, then by Proposition 2.12, we see that the integer $\rho < n+1$, therefore $\rho \leq n$. If $p^{n-1}(p-1)^2 \leq e < p^n(p-1)$, then

- we have $\lfloor \frac{p^i}{e} \rfloor = 0$ for all $0 \leq i \leq (n-1)$ as $p \neq 2$;
- similarly $\lfloor \frac{\lfloor \frac{e}{p-1} \rfloor + 1}{e} \rfloor \leq \lfloor \frac{1}{p-1} + \frac{1}{e} \rfloor = 0$, as we have assumed that $e > (p-1)$.

Therefore by Proposition 2.5, we have that $\rho \leq n$.

Now in case $p = 2$, the relevant formulas simplify. When $2^{n-1} < e < 2^n$, we get $\rho \leq (n+1)$ by Proposition 2.5. When $e = 2^{n-1}$, we get $\rho \leq (n+1)$ by Proposition 2.12. $\square$

Let us summarize the outcome of the previous two subsections.

Notation 2.14. For each pair of positive integers (e, j), we denote

$$d(e,j) \coloneqq \min\left(\sum_{i=1}^{n-1} \lfloor \frac{p^i}{e} \rfloor + \lfloor \frac{\lfloor \frac{e \cdot j}{p-1} \rfloor + 1}{e} \rfloor + nj, (\frac{\lfloor \frac{e \cdot j}{p-1} \rfloor}{p^{n-1}(p-1)} + n) \cdot j \right),$$

where n is the smallest natural number such that $e \cdot j < p^n(p-1)$.

Proposition 2.15. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. Then we have $\rho \leq d(e,j)$.*

Proof. Combine Proposition 2.5 and Proposition 2.12. $\square$

2.4. **Argument for boundedness.** Lastly let us show that an additional condition gives rise to boundedness of length of $\mathfrak{S}/J$.

Proposition 2.16. *Let $J \subset \mathfrak{S}$ and $j \in \mathbb{N}$ be as in Situation 2.1. Assume moreover that there is an $\ell \in \mathbb{N}$ such that $E^\ell \cdot \varphi(J) \subset J$, then $p^{(\rho+\max(j,\ell))\cdot\sigma} \in J$. The additional assumption implies that* $\mathrm{length}(\mathfrak{S}/J) \leq$ *$(\rho + \max(j,\ell)) \cdot \sigma^2$, in particular $(u,p)^{(\rho+\max(j,\ell))\cdot\sigma^2} \subset J$.*

Proof. For any ideal $I \subset \mathfrak{S}$, we denote $(I : p) \coloneqq \{f \in \mathfrak{S} \mid p \cdot f \in I\}$. Alternatively, the ideal is defined via the following exact sequence:

$$0 \to (I : p) \to \mathfrak{S} \xrightarrow{\cdot p} \mathfrak{S}/I.$$

Since (E, p) is a regular sequence, one checks that $E \cdot (I : p) = (E \cdot I : p)$. Using the fact that φ is flat, one checks that $\varphi(I : p) = (\varphi(I) : p)$. Therefore if we let $J_0 = J$ and inductively define $J_i = (J_{i-1} : p)$ for all $i \geq 1$, then we can make the following observations:

(1) We have $\mathfrak{S}/J_n \xrightarrow[\cong]{\cdot p^n} p^n \cdot \mathfrak{S}/J$, hence $\mathfrak{S}/(J_n + (p)) \xrightarrow[\cong]{\cdot p^n} \frac{p^n \cdot \mathfrak{S}/J}{p^{n+1} \cdot \mathfrak{S}/J}$;

(2) The ideals J_n again satisfy conditions: $E^j \cdot (-) \subset \varphi(-) \cdot \mathfrak{S}$ and $E^\ell \cdot \varphi(-) \subset (-)$.

Our task is to show that $J_n = \mathfrak{S}$ when $n \geq (\rho + \max(j,\ell)) \cdot \sigma$. Letting σ_n and ρ_n be defined by $J_n + (p) = (p, u^{\sigma_n})$ and $J_n + (u) = (u, p^{\rho_n})$, it suffices to show that $\sigma_i - \sigma_{i+\rho+\max(j,\ell)} \geq 1$. Since ρ_n is non-increasing, using the observation (2) above, it suffices to prove the above with $i = 0$.

Suppose to the contrary we have $0 < \sigma_0 = \ldots = \sigma_{\rho+\max(j,\ell)}$, we need to deduce a contradiction. This assumption, together with the observation (1) above, implies that multiplication by p map on $A \coloneqq \mathfrak{S}/J$ induces isomorphisms:

$$A/pA \xrightarrow[\cong]{\cdot p} pA/p^2A \xrightarrow[\cong]{\cdot p} \ldots \xrightarrow[\cong]{\cdot p} p^{\rho+\max(j,\ell)}A/p^{\rho+\max(j,\ell)+1}A.$$

Weierstrass preparation and the definition of σ implies the existence of a polynomial $f \in J$ such that $f \equiv u^\sigma \mod p$. Since (f, p) is a regular sequence, one checks that the p-adic filtration on $B \coloneqq \mathfrak{S}/f$ also satisfies $B/pB \xrightarrow[\cong]{\cdot p} pB/p^2B \xrightarrow[\cong]{\cdot p} \ldots$. Let us now look at the map $\mathfrak{S}/(f, p^{\rho+\max(j,\ell)+1}) \twoheadrightarrow \mathfrak{S}/(J, p^{\rho+\max(j,\ell)+1})$, it is an isomorphism modulo p so, by the above knowledge of p-adic filtrations on both sides, it is an isomorphism. Therefore we have $J \equiv (f) \mod p^{\rho+\max(j,\ell)+1}$. Moreover the definition of ρ implies that the constant term of f must have p-adic valuation ρ. Now our conditions imply that there exists polynomials $P(u), Q(u) \in W/p^{\rho+\max(j,\ell)+1}[u]$ such that we have equalities

$$E(u)^j \cdot f = \varphi(f) \cdot P(u) \text{ and } E(u)^\ell \cdot \varphi(f) = f \cdot Q(u)$$

in $W/p^{\rho+\max(j,\ell)+1}[u]$. Now the constant term of left hand side of both equations are nonzero in $W/p^{\rho+\max(j,\ell)+1}$, therefore the Newton polygon of $E(u)^j \cdot f$ is the same as that of $\varphi(f)\cdot \widetilde{P}(u)$ where $\widetilde{P}(u) \in W[u]$ is an arbitrary lift of $P(u)$. Consequently we see that there is an inclusion of sets:

$$\{p\text{-adic valuations of roots of } \varphi(f)\} \subset \{p\text{-adic valuations of roots of } f\} \cup \{1/e\}.$$

Similarly we also have an inclusion of sets:

$$\{p\text{-adic valuations of roots of } f\} \subset \{p\text{-adic valuations of roots of } \varphi(f)\} \cup \{1/e\}.$$

Since we have an equality of subsets of $\mathbb{Q}$:

$$1/p \cdot \{p\text{-adic valuations of roots of } f\} = \{p\text{-adic valuations of roots of } \varphi(f)\},$$

we arrive at the following contradiction:

$$\{p\text{-adic valuations of roots of } f\} \cup \{1/e\} = (1/p \cdot \{p\text{-adic valuations of roots of } f\}) \cup \{1/e\}.$$

Therefore we see that we cannot have $(\rho + \max(j,\ell))$ many σ's being all equal, which finishes the proof. □

3. Some prismatic cohomology facts

In this section, we recall some statements concerning torsion in prismatic cohomology. Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$.

Remark 3.1. Recall (see [BMS18, Proposition 4.3] and [BS22, Theorem 1.8.6]) that the prismatic cohomology $\mathfrak{M}^i := \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, being a Breuil–Kisin module, admits the following canonical exact sequences:

$$0 \to \mathfrak{M}^i_{\mathrm{tors}} = \mathfrak{M}^i[p^\infty] \to \mathfrak{M}^i \to \mathfrak{M}^i_{\mathrm{tf}} \to 0,$$

$$0 \to \mathfrak{M}^i_{\mathrm{tf}} \to (\mathfrak{M}^i)^{\vee\vee} \to \overline{\mathfrak{M}^i} \to 0,$$

where $(\mathfrak{M}^i)^{\vee\vee}$ is the double dual (or reflexive hull) of $\mathfrak{M}^i$ which is finite free over $\mathfrak{S}$ and $\overline{\mathfrak{M}^i}$ is supported at the closed point (p,u) of $\mathrm{Spec}(\mathfrak{S})$.

The following result is the main reason why we studied the kind of ideal J in Situation 2.1.

Proposition 3.2. *Let $n \in \mathbb{N} \cup \{\infty\}$, denote $\mathfrak{M}^i_n := \mathrm{H}^i(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})/^L p^n)$ (where $n = \infty$ means that we do not perform the reduction at all). Then we have the following:*

1. *For all $i \geq 0$, there exists maps $F\colon \varphi^*_{\mathfrak{S}}\mathfrak{M}^i_n \to \mathfrak{M}^i_n$ and $V\colon \mathfrak{M}^i_n \to \varphi^*_{\mathfrak{S}}\mathfrak{M}^i_n$ such that both $F\circ V$ and $V \circ F$ are the same as multiplication by E^i;*
2. *For all $i > 0$, multiplication by E^{i-1} on $\varphi^*_{\mathfrak{S}}\mathfrak{M}^i_n$ factors through a submodule of $\mathfrak{M}^i_n$.*

In particular, when $i > 0$, let J be the annihilator ideal of $\mathfrak{M}^i_n[u^\infty]$. Then the ideal J and $(j,\ell) = (i-1,i)$ satisfy the conditions in Situation 2.1 and Proposition 2.16.

When $n=\infty$, the statement (1) follows from [BS22, Theorem 1.8.(6)]. In general, both (1) and (2) follow from the observation made in [LL23, Proposition 3.2]. For the convenience of the readers, let us sketch the argument below.

Proof. Recall that the Frobenius-twisted prismatic cohomology admits Nygaard filtrations, see [BS22, Section 15]. In particular, for any $j \geq 0$, there are natural maps $\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n) \to \varphi^*_{\mathfrak{S}}\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$ and $\varphi^*_{\mathfrak{S}}\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n) \to \mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n)$ such that compositions either way are the same as multiplication by E^j. Moreover these Nygaard filtrations admit divided Frobenius maps to prismatic cohomology: $\mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n) \xrightarrow{\varphi_j} \mathrm{R}\Gamma(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$.

By [LL25, Lemma 7.8.(3)], the induced map $\mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^j/p^n) \xrightarrow{\varphi_j} \mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$ is an isomorphism. This gives (1) by considering i-th Nygaard filtration. Also by [LL25, Lemma 7.8.(3)], when $j>0$, the induced map $\mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \mathrm{Fil_N}^{j-1}/p^n) \xrightarrow{\varphi_{j-1}} \mathrm{H}^j(\mathcal{X}_{\mathrm{qsyn}}, \Delta/p^n)$ is injective. This gives (2) by considering $(i-1)$-st Nygaard filtration. The last sentence is a consequence of (1) and (2). □

Remark 3.3. Let us take the opportunity to correct an error in [LL25, Lemma 7.8.(3)]. The proof has a gap in its last sentence: namely, when we use the same proof strategy to run the argument for proving the derived mod p^m versions, the cohomological estimate might be off by 1 cohomological degree due to p-torsion in $\Omega^{i+1}_{X/(A/I)}$, and this p-torsion subsheaf is nonzero exactly when A/I contains p-torsion (and $X/(A/I)$ has relative dimension at least $i+1$). Therefore, by the proof strategy of loc. cit. we get the following conclusion: The statement of [LL25, Lemma 7.8.(1)-(3)] is correct as is, but for their derived mod p^m analogs, one needs an extra assumption that (A, I) is a transversal prism (namely A/I is p-torsion free). So, one just needs to change the last sentence to "Moreover their derived mod p^m counterparts hold *as long as (A, I) is transversal.* Fortunately, the Breuil–Kisin prism is an example of such, which justifies our usage of [LL25, Lemma 7.8.(3)] in the above proof. Lastly we point out that in the proof of [LL25, Lemma 7.8.(3)], the authors give a reference to [BS22, Theorem 15.2.(2)] for the cohomological estimate, but the more appropriate reference seems to be rather [BS22, Theorem 15.3].

The rest of this section concerns the A_{inf} cohomology defined in [BMS18, Theorem 1.8], let us recall some key definitions and properties below.

Notations 3.4. Let C be the completion of an algebraic closure of K, with its tilt $C^\flat$ defined as follows: Consider the ring of integers $\mathcal{O}_C \subset C$, then define $\mathcal{O}_C^\flat := \lim_\varphi(\mathcal{O}_C/p)$. Given a sequence of elements $\{x_i\}_{i\in\mathbb{N}}$ of $\mathcal{O}_C/p$ satisfying $x_i^p = x_{i-1}$, we denote by $\underline{x}$ its corresponding element in $\mathcal{O}_C^\flat$. It is a fact that $\mathcal{O}_C^\flat$ is a rank 1 valuation ring, whose fraction field $\mathrm{Frac}(\mathcal{O}_C^\flat) =: C^\flat$ is an algebraically closed complete non-archimedean field of equal characteristic p. The maximal ideal of $\mathcal{O}_C^\flat$ is given by $\mathfrak{m}_C^\flat = \{\underline{x} \in \mathcal{O}_C^\flat \mid x_0 \in \mathfrak{m}_C/(p\cdot\mathcal{O}_C) \subset \mathcal{O}_C/p\}$. For more on this, we refer readers to [Sch12, Section 3].

Fix a choice of compatible p-power primitive roots of unity $(1, \zeta_p, \zeta_{p^2}, \cdots)$, then the sequence $\{\zeta_{p^i}\}_{i\in\mathbb{N}}$ defines an element $\epsilon \in \mathcal{O}_C^\flat$. The Fontaine period ring A_{inf} is defined as the (p-typical) Witt ring of $\mathcal{O}_C^\flat$, equipped with Frobenius automorphism φ. The following two elements $\mu := [\epsilon] - 1$ and $\widetilde{\xi} = \varphi(\xi) = \frac{\varphi(\mu)}{\mu}$ in A_{inf} are important to us.

In the rest of this section C can be any algebraically closed complete non-archimedean field of mixed characteristic $(0, p)$.

Remark 3.5. Let $\mathfrak{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_C)$ with its rigid generic fiber $X := \mathfrak{X}_C$. There is a natural map of sites $\nu\colon X_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{Zar}}$, then according to [BMS18, Definition 8.1 and 9.1], one defines

$$A\Omega_{\mathfrak{X}} := L\eta_\mu(R\nu_*\mathbb{A}_{\mathrm{inf},X}) \text{ and } \widetilde{\Omega}_{\mathfrak{X}} := L\eta_\mu(R\nu_*\mathbb{A}_{\mathrm{inf},X}/\widetilde{\xi}).$$

For the purpose of this paper, we merely view the above as objects in $D(\mathfrak{X}_{\mathrm{Zar}}, A_{\mathrm{inf}})$. The A_{inf} cohomology is then defined as

$$\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := \mathrm{R}\Gamma(\mathfrak{X}_{\mathrm{Zar}}, A\Omega_{\mathfrak{X}}).$$

By [BMS18, Theorem 1.8], all cohomology groups are Breuil–Kisin–Fargues modules (see [BMS18, Definition 4.22]). Analogous to Remark 3.1, using [BMS18, Proposition 4.13], we see that $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$ also admits a natural exact sequence:

$$0 \to M^i_{\mathrm{tors}} = M^i[p^{\gg 0}] \to M^i \to M^i_{\mathrm{free}} \to \overline{M^i} \to 0,$$

with all modules appearing above, regarded as A_{inf}-complexes, perfect.

In general, (derived) reduction modulo an element certainly does not commute with $L\eta$ with respect to another element. Therefore it is surprising to learn (see [BMS18, Theorem 9.2.(1)]) that the natural map $A\Omega_{\mathfrak{X}}/\widetilde{\xi} \to \widetilde{\Omega}_{\mathfrak{X}}$ is a quasi-isomorphism! In [Bha18], at least if we work at the level of almost mathematics with respect to $[\mathfrak{m}_C^\flat]$, one finds a conceptual proof for this fact.

Proposition 3.6 ([Bha18, Lemma 5.16 and Proposition 7.5])**.** *The natural map $A\Omega_{\mathfrak{X}}/\widetilde{\xi} \to \widetilde{\Omega}_{\mathfrak{X}}$ is an almost, with respect to $[\mathfrak{m}_C^\flat]$, isomorphism in $D(\mathfrak{X}_{\mathrm{Zar}}, A^a_{\mathrm{inf}})$.*

Let us sketch the proof for later use.

Sketch of proof in loc. cit. The Lemma 5.16 in loc. cit. provides such a natural map, as well as a criterion for when the map is an almost isomorphism: it suffices for the cohomology sheaves of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})/\mu$ to be almost $\widetilde{\xi}$-torsionfree. Since $\widetilde{\xi} = \frac{(\mu+1)^p-1}{\mu} = \mu^{p-1} + \ldots + p \cdot \mu + p \equiv p$ modulo μ, it is equivalent to these cohomology sheaves being almost p-torsionfree. This later claim follows from Theorem 4.14 and Lemma 7.1 in loc. cit. $\square$

The above admits a direct generalization.

Proposition 3.7. *Define* $\widetilde{\Omega}^{(n)}_{\mathfrak{X}} := L\eta_\mu(R\nu_*\mathbb{A}_{\mathrm{inf},X}/\widetilde{\xi}^n) \in D(\mathfrak{X}_{\mathrm{Zar}}, A_{\mathrm{inf}})$*. Then the natural map* $A\Omega_{\mathfrak{X}}/\widetilde{\xi}^n \to \widetilde{\Omega}^{(n)}_{\mathfrak{X}}$ *is an almost, with respect to* $[\mathfrak{m}^\flat_C]$*, isomorphism in* $D(\mathfrak{X}_{\mathrm{Zar}}, A^a_{\mathrm{inf}})$*.*

Proof. Using again [Bha18, Lemma 5.16], we are reduced to showing that the cohomology sheaves of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})/\mu$ are almost $\widetilde{\xi}^n$-torsionfree. Since this is equivalent to these sheaves being almost $\widetilde{\xi}$-torsionfree, we are done thanks to the proof of Proposition 3.6. $\square$

Lemma 3.8. *Set* $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$*, then there exists an* $N \gg 0$ *such that* $M^i[\widetilde{\xi}^\infty] = M^i[\widetilde{\xi}^N]$*. Moreover* $M^i[\widetilde{\xi}^\infty]$ *is a finitely presented coherent* A_{inf}*-module.*

Proof. By Remark 3.5, there exists an $m \in \mathbb{N}$ such that the torsion submodule in M^i is given by $M := M^i[p^m]$, which is a perfect complex. In particular, it is finitely presented. Using [BMS18, Lemma 3.26], we know that $W_m(\mathcal{O}^\flat_C)$ is a coherent ring. By [Sta25, Tag 05CX], we see that M is a coherent $W_m(\mathcal{O}^\flat_C)$-module. Therefore we are reduced to showing: if M is a finitely presented $W_m(\mathcal{O}^\flat_C)$-module, then there exists an $N \gg 0$ such that $M[\widetilde{\xi}^\infty] = M[\widetilde{\xi}^N]$. Indeed, we may then apply [Sta25, Tag 05CW] to see that $M[\widetilde{\xi}^N] = \ker(M \xrightarrow{\widetilde{\xi}^N} M)$ is a finitely presented coherent $W_m(\mathcal{O}^\flat_C)$-module.

Let us prove the above claim, by induction on the smallest power p^m of p that annihilates M. If M is annihilated by p, this follows from the fact that $\mathcal{O}^\flat_C$ is a rank one valuation ring. Since $W_m(\mathcal{O}^\flat_C)$ is a coherent ring, we know that both $Q := M[p]$ and $M/Q \cong \mathrm{Im}(M \xrightarrow{\cdot p} M)$ are finitely presented $W_m(\mathcal{O}^\flat_C)$-modules. By induction, if $m > 1$, we see that the $\widetilde{\xi}^\infty$-torsion parts in both Q and M/Q are annihilated by $\widetilde{\xi}^{N'}$ for some $N' \gg 0$. By the snake lemma, there is a natural exact sequence

$$0 \to Q[\widetilde{\xi}^\infty] \to M[\widetilde{\xi}^\infty] \to M/Q[\widetilde{\xi}^\infty].$$

One immediately sees that $M[\widetilde{\xi}^\infty]$ is annihilated by $\widetilde{\xi}^{2N'}$, hence we are done. $\square$

The following is inspired by the proof of [Min21, Lemma 5.1].

Proposition 3.9. *Let* $i > 0$ *and set* $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$*, then* $M^i[\widetilde{\xi}^\infty]$ *is almost, with respect to* $[\mathfrak{m}^\flat_C]$*, annihilated by* μ^{i-1}*. In particular, let* $J_{\mathrm{inf}} \subset A_{\mathrm{inf}}$ *be the annihilator of* $M^i[\widetilde{\xi}^\infty]$*, then we have an inclusion* $\mu^{i-1} \cdot [\mathfrak{m}^\flat_C] \subset J_{\mathrm{inf}}$*.*

Proof. Let n be an arbitrary positive integer. Recall [BMS18, Corollary 6.5] that the $L\eta$ functor commutes with canonical truncation. Applying [BMS18, Lemma 6.9], we see that there is a commutative diagram in $D(\mathfrak{X}_{\mathrm{Zar}}, A^a_{\mathrm{inf}})$:

$$\begin{tikzcd}
\tau^{\leq(i-1)} A\Omega_{\mathfrak{X}} \arrow[r] \arrow[d, bend right, "f_1"'] & \tau^{\leq(i-1)} \widetilde{\Omega}^{(n)}_{\mathfrak{X}} \arrow[d, bend right, "f_2"'] \\
\tau^{\leq(i-1)} R\nu_*(\mathbb{A}_{\mathrm{inf},X}) \arrow[r] \arrow[u, bend right, "g_1"'] & \tau^{\leq(i-1)} R\nu_*(\mathbb{A}_{\mathrm{inf},X}/\widetilde{\xi}^n), \arrow[u, bend right, "g_2"']
\end{tikzcd}$$

where both horizontal arrows are induced by $\tau^{(i-1)}$ applied to the (derived) reduction modulo $\widetilde{\xi}^n$ map, and the composition of f_j and g_j in either direction is μ^{i-1} for $j = 1, 2$.

By [Sch13, Theorem 5.1 and proof of Theorem 8.4], we get almost isomorphisms

$$\mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \cong \mathrm{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf}}) \text{ and } \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/\widetilde{\xi}^n \cong \mathrm{R}\Gamma(X_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf}}/\widetilde{\xi}^n)$$

with respect to $[\mathfrak{m}_C^\flat]$. Now we take $(i-1)$-st cohomology of the diagram above, and arrive at the following commutative diagram of almost $A_{\inf}$-modules:

$$\begin{tikzcd} \mathrm{H}^{i-1}_{A_{\inf}}(\mathfrak{X}) \arrow[r] \arrow[d, bend right, "f_1"'] & \mathrm{H}^{i-1}(\mathfrak{X}, \widetilde{\Omega}^{(n)}_{\mathfrak{X}}) \cong \mathrm{H}^{i-1}(\mathrm{R}\Gamma_{A_{\inf}}(\mathfrak{X})/\widetilde{\xi}^n) \arrow[d, bend right, "f_2"'] \\ \mathrm{H}^{i-1}(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\inf} \arrow[r, two heads] \arrow[u, bend right, "g_1"'] & \mathrm{H}^{i-1}(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\inf}/\widetilde{\xi}^n, \arrow[u, bend right, "g_2"'] \end{tikzcd}$$

where the identification of top-right item uses Proposition 3.7, and the composition of f_j and g_j in either direction is μ^{i-1} for $j = 1, 2$. Since the cokernel of the top arrow is, as an almost $A_{\inf}$-module, given by $\mathrm{H}^i_{A_{\inf}}(\mathfrak{X})[\widetilde{\xi}^n]$, we see that $\mathrm{H}^i_{A_{\inf}}(\mathfrak{X})[\widetilde{\xi}^n]$ is almost annihilated by μ^{i-1}. By Lemma 3.8, we can choose n large enough so that $\mathrm{H}^i_{A_{\inf}}(\mathfrak{X})[\widetilde{\xi}^n] = \mathrm{H}^i_{A_{\inf}}(\mathfrak{X})[\widetilde{\xi}^\infty]$. □

Lemma 3.10. *Let R be a coherent ring, and let M be a finitely presented R-module. Then the annihilator ideal of M is finitely presented.*

Proof. Choose generators $x_i \in M$, each generates a finitely generated submodule $N_i := R \cdot x_i \subset M$. By [Sta25, Tag 05CX], the module M is coherent, hence the N_i's are all finitely presented. Hence we see that each x_i has a finitely generated annihilator ideal J_i. As R is coherent, they are automatically finitely presented. Finally, it suffices to show that the intersection of two finitely presented ideals in R is again finitely presented. This follows from applying [Sta25, Tag 05CW] to $J_1 \cap J_2 = \ker(J_1 \to R/J_2)$. □

Corollary 3.11. *With setup and notation as in Proposition 3.9. The ideal $J_{\inf} \subset A_{\inf}$ is a finitely generated ideal containing some power of p, therefore we in fact have $\mu^{i-1} \in J_{\inf}$.*

Proof. By Lemma 3.8 and its proof, we see that $M^i[\widetilde{\xi}^\infty]$ is a finitely presented $W_m(\mathcal{O}_C^\flat)$-module, and the ideal $J_{\inf}$ is the preimage under the projection $A_{\inf} \xrightarrow{\bmod p^m} W_m(\mathcal{O}_C^\flat)$ of the annihilator ideal $J' \subset W_m(\mathcal{O}_C^\flat)$ of $M^i[\widetilde{\xi}^\infty]$. Hence it suffices to know that J' is finitely presented, which follows from combining Lemma 3.8 and Lemma 3.10.

It remains to show that $\mu^{i-1} \in J'$, which is equivalent to $W_m(\mathcal{O}_C^\flat) = \ker(W_m(\mathcal{O}_C^\flat) \xrightarrow{\cdot\mu^{i-1}} W_m(\mathcal{O}_C^\flat)/J')$. Using [Sta25, Tag 05CW] we see that the kernel is a finitely generated ideal. By Proposition 3.9, we see this finitely generated ideal contains the image of $[\mathfrak{m}_C^\flat]$, therefore it must be the unit ideal. □

4. Applications

Throughout this section, let $\mathcal{X}$ be a smooth proper formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$. In this section, we deduce consequences of the previous sections. We begin with an auxilliary lemma.

Lemma 4.1. *Let C be a complete algebraically closed nonarchimedean extension of $\mathbb{Q}_p$. Let $v_{C^\flat}$ be the valuation on the tilt $C^\flat$, normalized so that $v_{C^\flat}(p^\flat) = 1$. Let $j > 0$ and consider the Teichmüller expansion*

$$\mu^j = \sum_{i \geq 0} p^i \cdot [x_i^{(j)}] \in W(\mathcal{O}_C^\flat),$$

then we have $v_{C^\flat}(x_{j\ell}^{(j)}) = j \cdot \frac{p}{p^\ell(p-1)}$ for any $\ell \in \mathbb{N}$.

Proof. Recall that the addition in (p-typical) Witt vectors of a perfect ring R is defined in the following manner. First there are universal polynomials $Q_i(X, Y) \in \mathbb{Z}[X, Y]$ defined inductively by

$$X^{p^n} + Y^{p^n} = \sum_{i=0}^{n} p^i Q_i^{p^{n-i}}.$$

Then we have

$$[x] + [y] = \sum_{i \geq 0} p^i \cdot [Q_i(x^{1/p^i}, y^{1/p^i})] \text{ in } W(R)$$

for any $x, y \in R$. We can inductively see that

- Each $Q_i(X,Y)$ is a homogeneous degree p^i polynomial;
- $Q_0(X,Y) = X+Y$;
- whenever $i > 0$ there is an expansion of the form $Q_i(X,Y) = \sum_{1\leq m\leq p^i-1} a_m X^m Y^{p^i-m}$ with $a_1 = a_{p^i-1} = 1$.

For $x_i := x_i^{(1)}$, we have from the above two expansions

$$[\epsilon] + \sum_{i>0} p^i[Q_i((\epsilon-1)^{1/p^i}, 1)] = [\epsilon - 1] + 1 = [\epsilon] + (-1)\cdot\sum_{i>0} p^i[x_i],$$

and $x_0 = \epsilon - 1$. In particular, we see that

$$(-1)\cdot\sum_{i>0} p^i[Q_i((\epsilon-1)^{1/p^i}, 1)] = \sum_{i>0} p^i[x_i].$$

We claim that our lemma follows from this equality, together with the discussions of "Newton polygon" in [FF18, Subsection 1.5].

Let us first summarize necessary definitions and facts concerning Newton polygons: In [FF18, Definition 1.5.2], to any element $y = \sum_{i\geq 0} p^i\cdot[y_i] \in A_{\mathrm{inf}}$, the authors define $\mathcal{N}ewt(y)$ to be the function $\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ whose graph is the highest convex non-increasing polygon below the points $\{(n, v_{C^\flat}(y_n)) \mid n\in\mathbb{N}\}$. By how $\mathcal{N}ewt(y)$ is defined, we see that if (n, v_n) is a turning point of its graph, then $v_{C^\flat}(y_n) = v_n$. On [FF18, p. 20], the authors conclude that $\mathcal{N}ewt(y\cdot z) = \mathcal{N}ewt(y) * \mathcal{N}ewt(z)$, where the operation $*$ of convex functions is defined on [FF18, p. 18]. Using this, one checks that $\mathcal{N}ewt(u\cdot y) = \mathcal{N}ewt(y)$ if u is a unit.

Now we are ready to prove the claim for $j = 1$: Using the previous paragraph, we see that

$$\mathcal{N}ewt(\sum_{i>0} p^i[x_i]) = \mathcal{N}est(\sum_{i>0} p^i[Q_i((\epsilon-1)^{1/p^i}, 1)]).$$

By the third observation on these Q_i's, we have $v_{C^\flat}(Q_i((\epsilon-1)^{1/p^i}, 1)) = \frac{p}{p^i(p-1)}$ for all $i > 0$. So the Newton polygon goes precisely through $\{(n, \frac{p}{p^{n-1}(p-1)}) \mid n\in\mathbb{N}\}$ for all $n\geq 1$, and these points are all turning points. In the end we deduce that $v_{C^\flat}(x_i) = \frac{p}{p^i(p-1)}$ for all $i > 0$ as well.

The $j = 1$ case implies the general case, as follows: Chasing through the definition of $*$, the graph of $\mathcal{N}ewt(y^j)$ is the original graph of $\mathcal{N}ewt(y)$ scaled by j-times. Therefore the turning points of $\mathcal{N}ewt(\mu^j)$ are given by $\{(j\cdot n, j\cdot\frac{p}{p^{n-1}(p-1)}) \mid n\in\mathbb{N}\}$, finishing the proof. $\square$

With the above preparation, we can prove the following.

Theorem 4.2. *Let $i > 0$, denote $\mathfrak{M}^i := \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, and let J be the annihilator ideal of $\mathfrak{M}^i_n[u^\infty]$. Let ρ be defined by $J + (u) = (u, p^\rho)$. If $e\cdot(i-1) < p^n(p-1)$, then $\rho \leq (i-1)\cdot n$.*

By [LL23, Corollary 3.8 or Remark 3.9], the $\mathfrak{M}^1$ is always finite free. Therefore in the following proof, we always assume that $i\geq 2$, hence $(i-1) > 0$. So we may summon Lemma 4.1 for $j = (i-1)$.

Proof. Let $\mathfrak{X} := \mathcal{X}_{\mathcal{O}_C}$, and set $M^i := \mathrm{H}^i(\mathrm{R}\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}))$. After choosing compatible p-power roots of π in $\mathcal{O}_C$, we get an element $\pi^\flat\in\mathcal{O}_C^\flat$ (see Notations 3.4). We may consider the map of prisms which is p-completely faithfully flat:

$$f\colon (\mathfrak{S} = W[\![u]\!], (E)) \to (A_{\mathrm{inf}}, (\xi)),$$

with $f(u) = [\pi^\flat]$. By [BS22, Theorem 1.8.(5) and Theorem 17.2], we get a canonical isomorphism $\mathfrak{M}^i\otimes_{\mathfrak{S},\varphi\circ f} A_{\mathrm{inf}} \cong M^i$. Using the p-completely flatness of f, together with structural results mentioned in Remark 3.1 and Remark 3.5, we also get $\mathfrak{M}^i[u^\infty]\otimes_{\mathfrak{S},\varphi\circ f} A_{\mathrm{inf}} \cong M^i[\widetilde{\xi}^\infty]$. In particular, using again the p-completely flatness of f, the annihilator ideal J_{inf} of $M^i[\widetilde{\xi}^\infty]$ is given by $(\varphi\circ f)(J)\cdot A_{\mathrm{inf}}$.

Now suppose that $\rho > (i-1)\cdot n$, then we have $J\subset (u, p^{(i-1)\cdot n+1})$, consequently $J_{\mathrm{inf}}\subset J'_{\mathrm{inf}} := ([(\pi^\flat)^p]), p^{(i-1)\cdot n+1})$. Notice that an element $x = \sum_{m\geq 0} p^m\cdot[x_m]\in A_{\mathrm{inf}}$ lies in J'_{inf} if and only if $v_{C^\flat}(x_m)\geq\frac{p}{e}$ for all $m\leq (i-1)\cdot n$.

Corollary 3.11 says that $\mu^{i-1} \in J_{\inf}$. Therefore by the above paragraph, in the Teichmüller expansion of $\mu^{i-1} = \sum_{m \geq 0} p^m \cdot [x_m^{(i-1)}]$, we must have $v_{C^\flat}(x_m^{(i-1)}) \geq \frac{p}{e}$ for all $m \leq (i-1) \cdot n$. On the other hand, by Lemma 4.1 we have

$$v_{C^\flat}(x_{(i-1)\cdot n}^{(i-1)}) = (i-1) \cdot \frac{p}{p^n(p-1)}$$

contradicting with the assumption $e \cdot (i-1) < p^n(p-1)$. □

In practice, it is also important to understand the cohomology of $\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})/^L p^n)$. The above proof no longer works in this generality, but we have arguments purely from commutative algebra, at the expense of getting slightly worse bound.

Theorem 4.3. *Let $n \in \mathbb{N} \cup \{\infty\}$ and let $i > 0$, denote $\mathfrak{M}_n^i := \mathrm{H}^i(\mathrm{R\Gamma}_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})/^L p^n))$ (where $n = \infty$ means that we do not perform the reduction at all), and let J be the annihilator ideal of $\mathfrak{M}_n^i[u^\infty]$. Lastly, let σ and ρ be defined by $J + (p) = (p, u^\sigma)$ and $J + (u) = (u, p^\rho)$, we have*

(1) inequalities $\sigma \leq \lfloor \frac{e\cdot(i-1)}{p-1} \rfloor$ and $\rho \leq d(e, i-1)$;
(2) a belonging $p^{(\rho+i)\cdot\sigma} \in J$; and
(3) an inclusion $(u, p)^{(\rho+i)\cdot\sigma^2} \subset J$.

Proof. Using Proposition 3.2, the statement (1) follows from Proposition 2.15, the statement (2) follows from Proposition 2.16, whereas the statement (3) follows from the combination of (1) and (2). □

In [LL23] one finds results relating pathologies in p-adic geometry with u-torsion in prismatic cohomology, here let us update the conclusions with our new estimates.

Proposition 4.4. *Assume that the formal scheme $\mathcal{X}$ has an $\mathcal{O}_K$-point. Let $f \colon \mathrm{Alb}(\mathcal{X}_k) \to \mathrm{Alb}(\mathcal{X}_K)_k$ be the natural map discussed in the beginning of* [LL23, Subsection 4.1]. *Then $\ker(f)$ is p^n-torsion if $e < p^n(p-1)$.*

Proof. This follows from combination of Theorem 4.2, [LL23, Proposition 4.1] and [LL23, Theorem 4.2]. □

Proposition 4.5. *Let C be the completion of an algebraic closure of K, let $n \in \mathbb{N} \cup \{\infty\}$ and let $i > 0$, consider the specialization map*

$$\mathrm{Sp}_n^i \colon \mathrm{H}^i_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}/p^n) \to \mathrm{H}^i_{\acute{e}t}(\mathcal{X}_C, \mathbb{Z}/p^n)$$

discussed in the beginning of [LL23, Subsection 4.2] *(here again $n = \infty$ means that we do not perform reduction at all). Then $\ker(\mathrm{Sp}_n^i)$ is $p^{d(e,i-1)}$-torsion.*

Proof. This follows from Theorem 4.3 and [LL23, Theorem 4.14]. □

From now on, we use the notation from Remark 3.1. Let us observe that one can control $\overline{\mathfrak{M}^i}$ in terms of $\mathfrak{M}^i/p^N$ for some $N \gg 0$.

Lemma 4.6. *Let $\mathfrak{M}$ be any finitely generated $\mathfrak{S}$-module admitting exact sequences as in Remark 3.1, let p^m be such that it annihilates both $\mathfrak{M}_{\mathrm{tors}}$ and $\overline{\mathfrak{M}}$, then there is an exact sequence:*

$$0 \to \mathfrak{M}_{\mathrm{tors}} \oplus \overline{\mathfrak{M}} \to \mathfrak{M}/p^N \to (\mathfrak{M})^{\vee\vee}/p^N \to \overline{\mathfrak{M}} \to 0,$$

for any $N \geq 2m$. In particular, there is an identification $\mathfrak{M}/p^N[u^\infty] \simeq \mathfrak{M}[u^\infty] \oplus \overline{\mathfrak{M}}$ whenever $N \geq 2m$.

Proof. For any natural number n, we have canonical exact sequences:

$$0 \to \mathfrak{M}_{\mathrm{tors}}/p^n \to \mathfrak{M}/p^n \to \mathfrak{M}_{\mathrm{tf}}/p^n \to 0,$$

$$0 \to \overline{\mathfrak{M}}[p^n] \to \mathfrak{M}_{\mathrm{tf}}/p^n \to (\mathfrak{M})^{\vee\vee}/p^n \to \overline{\mathfrak{M}}/p^n \to 0.$$

The second sequence implies that $\mathfrak{M}_{\mathrm{tf}}/p^n[u^\infty] \cong \overline{\mathfrak{M}}[p^n]$, with the natural transitions map from $(n+1)$-st level to n-th level on the left hand side identified with the multiplication by p map on the right hand side.

Now let us denote $\mathfrak{N}_n \coloneqq \{x \in \mathfrak{M}/p^n \mid u^{\gg 0}x \in \mathfrak{M}_{\mathrm{tors}}/p^n \subset \mathfrak{M}/p^n\}$, then we have a canonical isomorphism $(\mathfrak{M}/p^n)/\mathfrak{N}_n \cong (\mathfrak{M}_{\mathrm{tf}}/p^n)/\overline{\mathfrak{M}}[p^n]$ and commutative diagrams of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{M}_{\mathrm{tors}}/p^{n+1} \arrow[r] \arrow[d, "\bmod p^n"'] & \mathfrak{N}_{n+1} \arrow[r] \arrow[d] & \overline{\mathfrak{M}}[p^{n+1}] \arrow[r] \arrow[d, "\cdot p"] & 0 \\
0 \arrow[r] & \mathfrak{M}_{\mathrm{tors}}/p^n \arrow[r] & \mathfrak{N}_n \arrow[r] & \overline{\mathfrak{M}}[p^n] \arrow[r] & 0.
\end{tikzcd}$$

If we consider the transition map from the N-th level to the m-th level, we get a splitting $\mathfrak{N}_N \simeq \mathfrak{M}_{\mathrm{tors}} \oplus \overline{\mathfrak{M}}$. The two sequences in the beginning combine into

$$0 \to \mathfrak{N}_n \to \mathfrak{M}/p^n \to (\mathfrak{M})^{\vee\vee}/p^n \to \overline{\mathfrak{M}}/p^n \to 0.$$

This finishes the proof. □

Corollary 4.7. *Let J' be the annihilator of $\overline{\mathfrak{M}^i}$ with $i > 0$, and let ρ' be such that $J' + (u) = (u, p^{\rho'})$. Then we have $\rho' \leq d(e, i-1)$.*

Proof. Since we have a natural injection $\mathfrak{M}^i/p^n \hookrightarrow \mathfrak{M}^i_n$, this follows from Lemma 4.6 and Theorem 4.3. □

Lastly we present our ultimate application:

Theorem 4.8. *There exists a constant $c(e, i)$ depending only on ramification index e and cohomological degree $i > 0$, such that if the $\mathrm{H}^i_{\acute{e}t}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}}$ is annihilated by p^m, then the $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W)_{\mathrm{tors}}$ is annihilated by p^{m+c}.*

Proof. By [BS22, Theorem 1.8.(1)&(5)], we have a natural exact sequence:

$$0 \to \mathfrak{M}^i/u \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_k/W) \otimes_{W,\varphi^{-1}} W \to \mathfrak{M}^{i+1}[u] \to 0.$$

By Theorem 4.3, we see that the third term is annihilated by $p^{d(e,i)}$. We claim that $\left(\mathfrak{M}^i/u\right)_{\mathrm{tors}}$ is annihilated by $p^{m+2\cdot d(e,i-1)}$. Our theorem follows from this claim, by taking $c = 2 \cdot d(e, i-1) + d(e, i)$.

To see our claim: By Remark 3.1, there is a natural exact sequence:

$$0 \to \mathfrak{M}^i_{\mathrm{tors}}/u \to \left(\mathfrak{M}^i/u\right)_{\mathrm{tors}} \to \left(\mathfrak{M}^i_{\mathrm{tf}}/u\right)_{\mathrm{tors}} \cong \overline{\mathfrak{M}^i}[u] \to 0.$$

By Corollary 4.7, we see that the third term above is annihilated by $p^{d(e,i-1)}$. We have reduced our claim to: the $\mathfrak{M}^i_{\mathrm{tors}}/u$ is annihilated by $p^{m+d(e,i-1)}$.

We have an exact sequence:

$$0 \to \mathfrak{M}^i[u^\infty] \to \mathfrak{M}^i_{\mathrm{tors}} \to \mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}} \to 0,$$

hence the following exact sequence:

$$0 \to \mathfrak{M}^i[u^\infty]/u \to \mathfrak{M}^i_{\mathrm{tors}}/u \to \mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}}/u \to 0.$$

By Theorem 4.3 (with $n = \infty$), we see that the first term above is annihilated by $p^{d(e,i-1)}$. Lastly by combining [BS22, Theorem 1.8.(5) and Section 17] and [BMS18, Theorem 1.8.(iv)], we see that there is a (non-canonical) isomorphism $\mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}}[1/u] \simeq \mathrm{H}^i_{\acute{e}t}(\mathcal{X}_C, \mathbb{Z}_p)_{\mathrm{tors}} \otimes_{\mathbb{Z}_p} \mathfrak{S}[1/u]$, therefore $\mathfrak{M}^i_{\mathrm{tors},u-\mathrm{tf}}$ is annihilated by p^m, finishing our proof. □

In the above, one could improve the constant c slightly by replacing the bound obtained in Theorem 4.3 with Theorem 4.2 at several places. However we feel that the constant c obtained via this method is unlikely to be optimal anyway, so we do not choose to optimize the bound in the proof to prevent complicating notations.

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O. Gabber: IHÉS, 35 route de Chartres, 91440 Bures-sur-Yvette, France
Email address: `gabber@ihes.fr`

S. Li: Morningside Center of Mathematics and State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email address: `lishizhang@amss.ac.cn`